Moment of Inertia of an Area About an Axis

Steven Vukazich
San Jose State University

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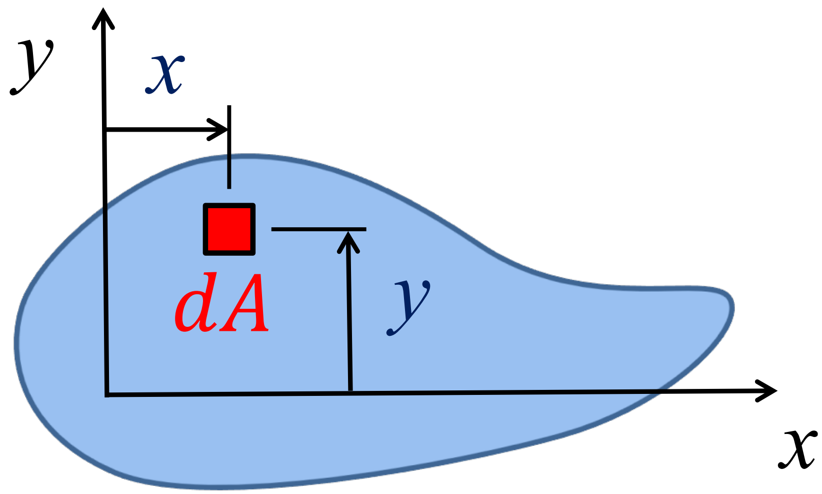


Recall we used the first moment of the area about an axis to find the centroid. The **Moment of Inertia** is the second moment of the area about an axis

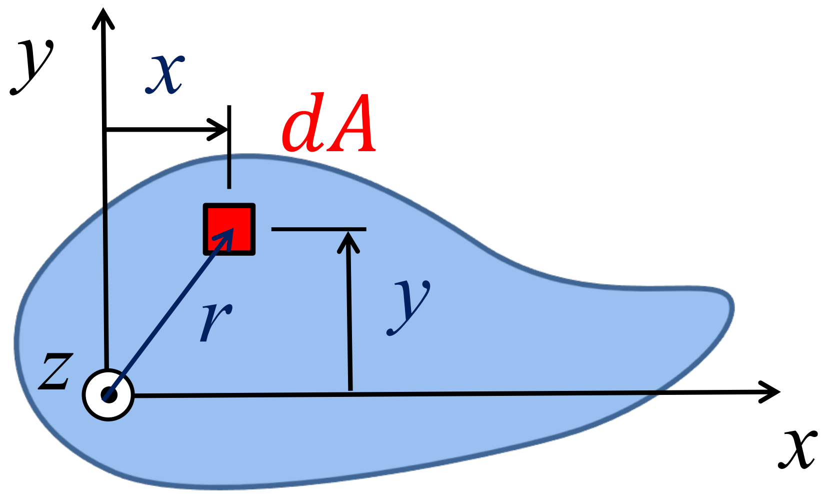
$$I_y = \iint x^2 dA$$

Second moment of the area about the *y* axis

$$I_x = \iint y^2 dA$$

Second moment of the area about the *x* axis

Polar Moment of Inertia



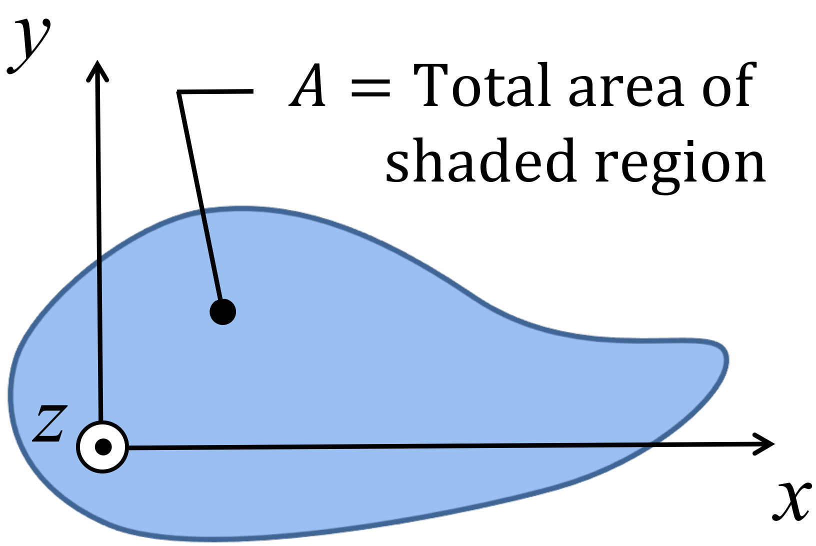
Polar Moment of Inertia
Second moment of the area about the *z* axis

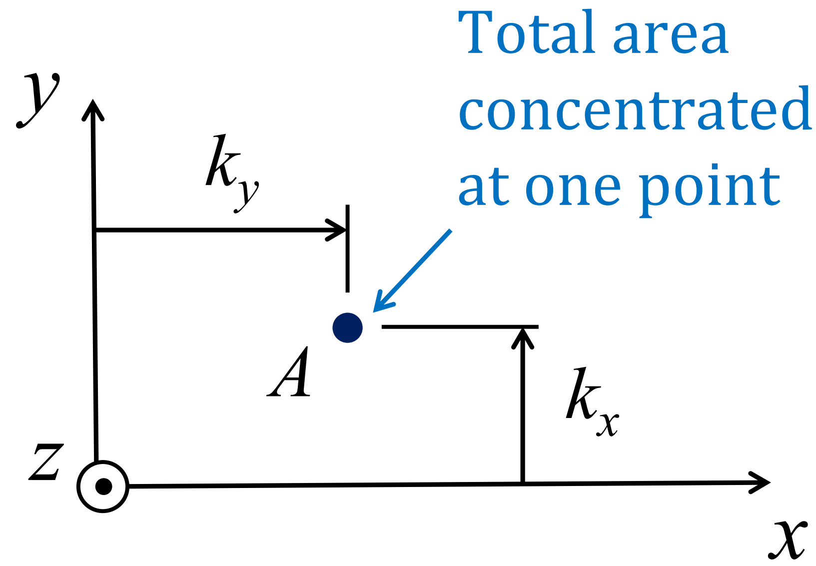
$$I_z = J_O = \iint r^2 dA$$

$$r^2 = x^2 + y^2$$

$$J_O = \iint (x^2 + y^2) dA = I_x + I_y$$

Radius of Gyration





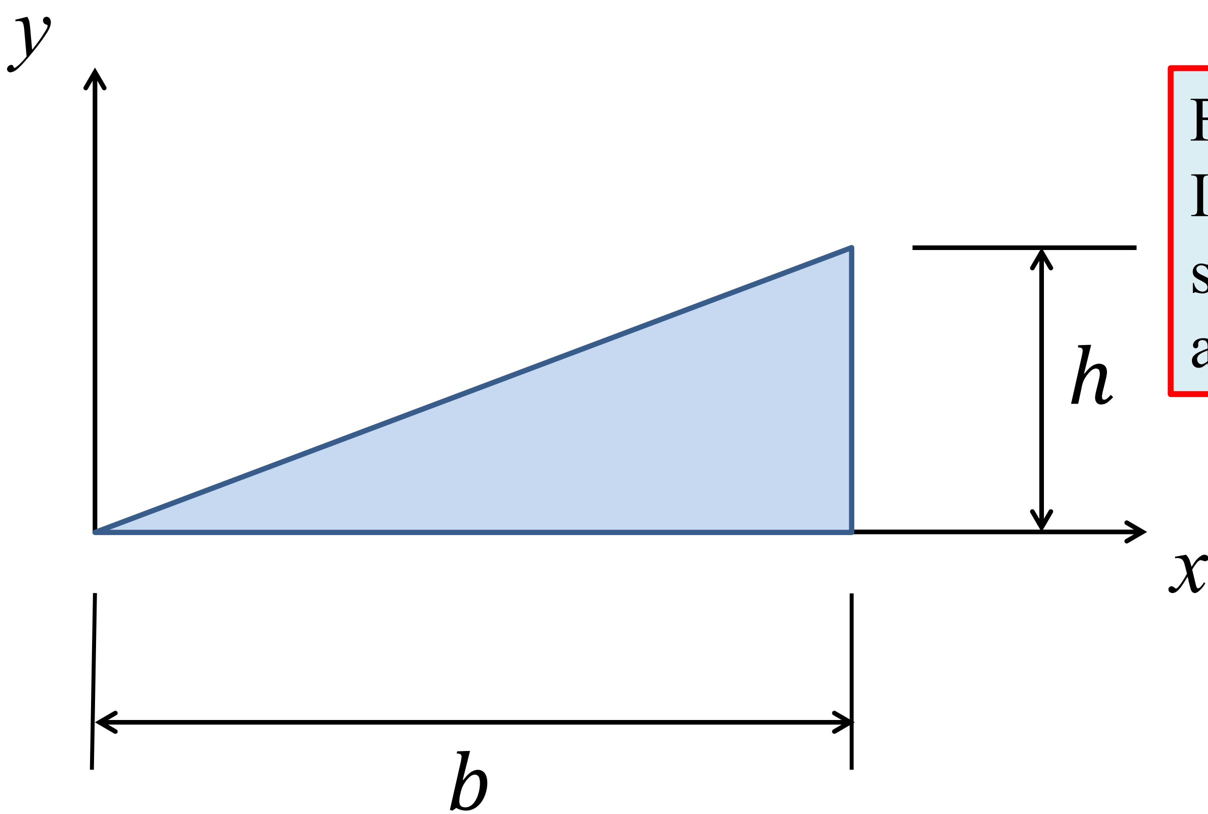
$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

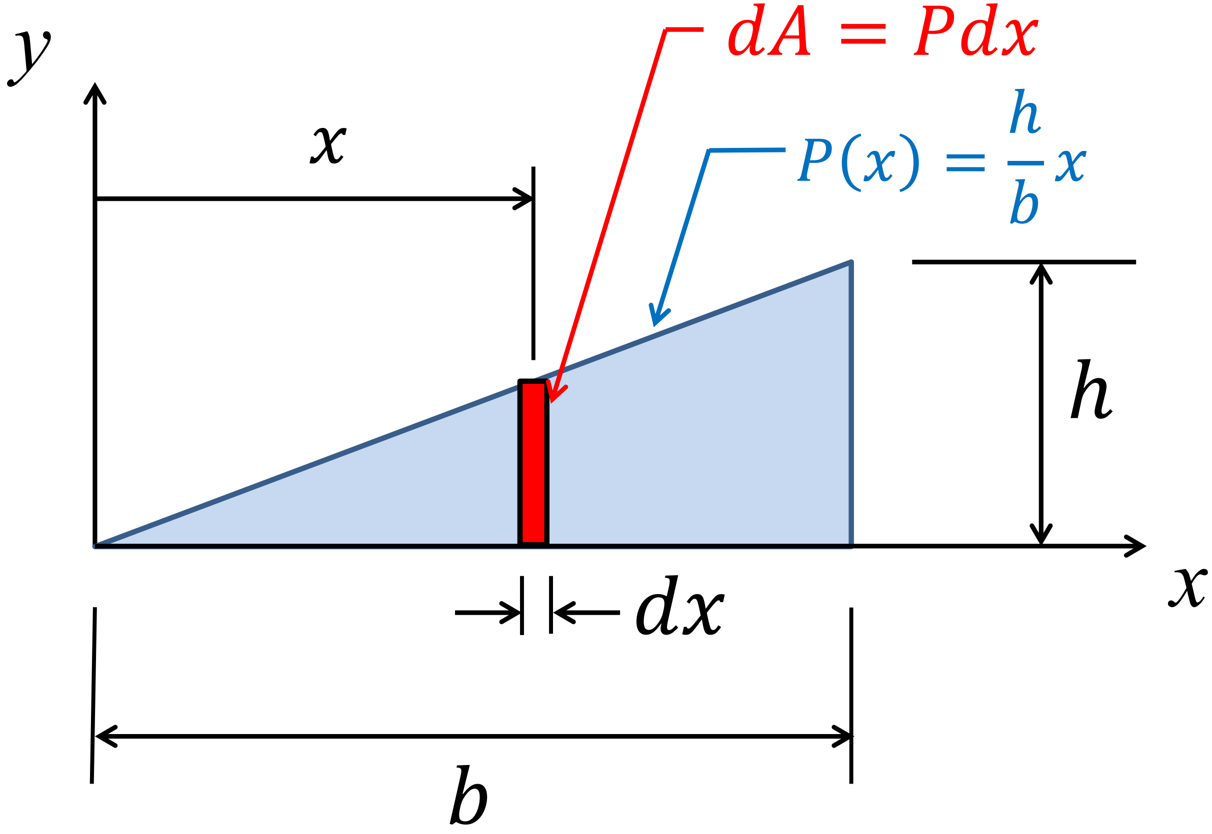
$$k = \sqrt{\frac{I}{A}}$$

Example Problem



Find the Moment of Inertia of the of the shaded area about the *x* and *y* axes shown.

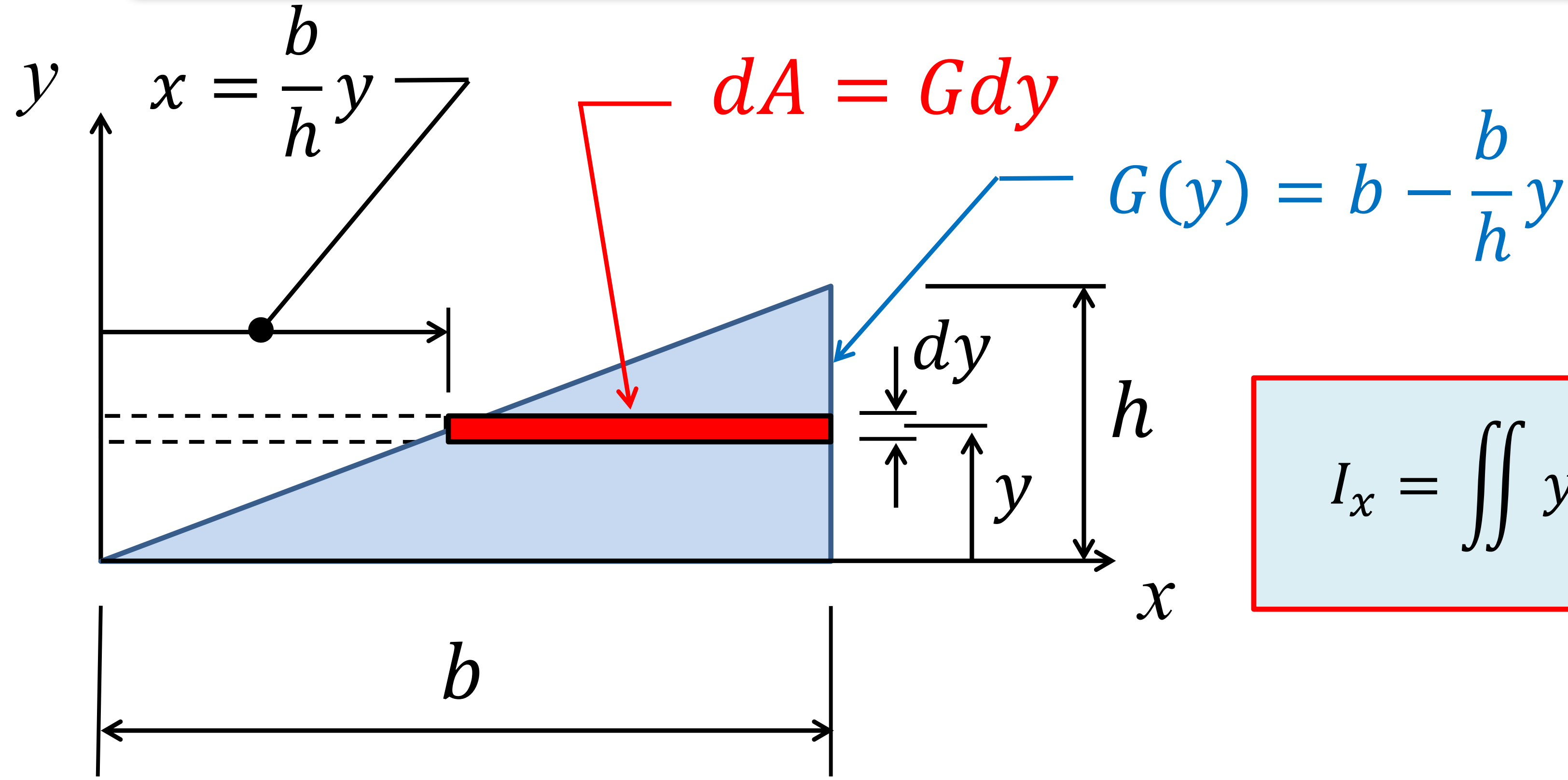
Divide the Area into Vertical Strips



$$I_y = \iint x^2 dA$$

$$I_y = \int_0^b x^2\, Pdx = \int_0^b x^2 \left(\frac{h}{b}\right) xdx = \frac{h}{b}\int_0^b x^3\, dx = \frac{h}{b}\left[\frac{x^4}{4}\right]_0^b = \frac{1}{4}b^3h$$

Divide the Area into Horizontal Strips

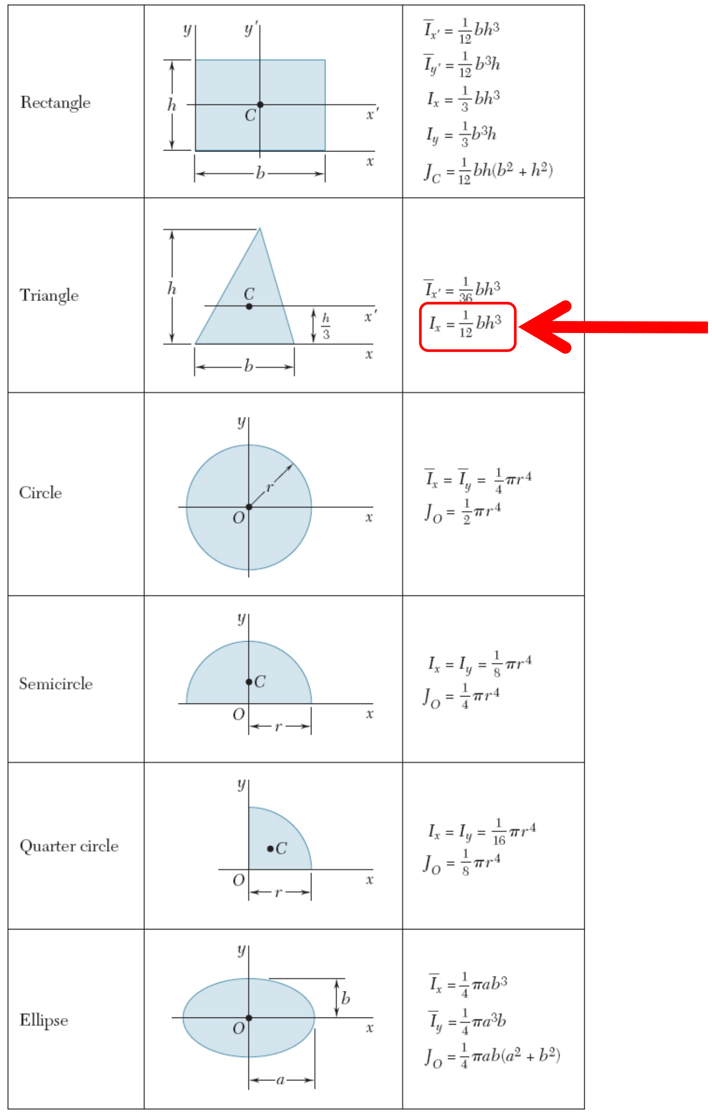


$$I_x = \iint y^2 dA$$

$$I_x = \int_0^h y^2\, Gdy = \int_0^h y^2 \left(b - \frac{b}{h}y\right) dy = b \int_0^h y^2\, dx - \frac{b}{h} \int_0^h y^3\, dx$$

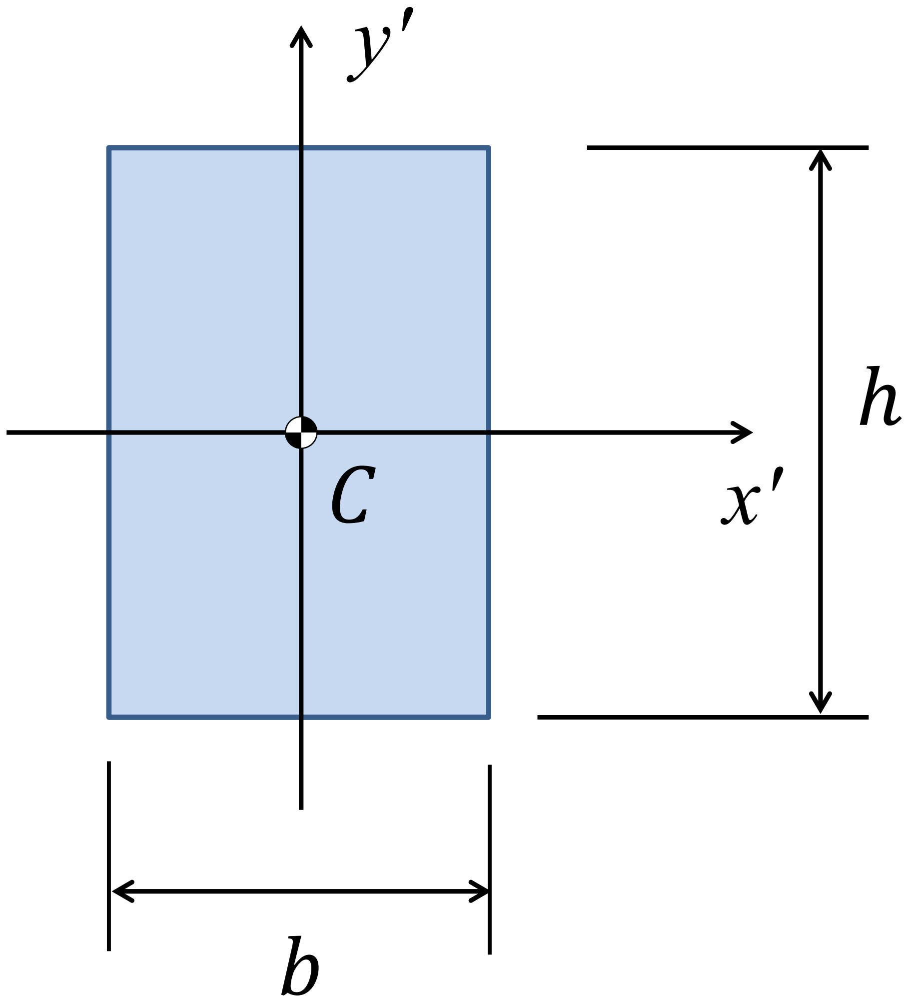
$$I_x = b\left[\frac{y^3}{3}\right]_0^h - \frac{b}{h}\left[\frac{y^4}{4}\right]_0^h = \frac{1}{3}bh^3 - \frac{1}{4}bh^3 = \frac{1}{12}bh^3$$

Result Agrees with the Tabulated Value for a General Triangular Area in Textbook



Rectangle	$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle	$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle	$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle	$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle	$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse	$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

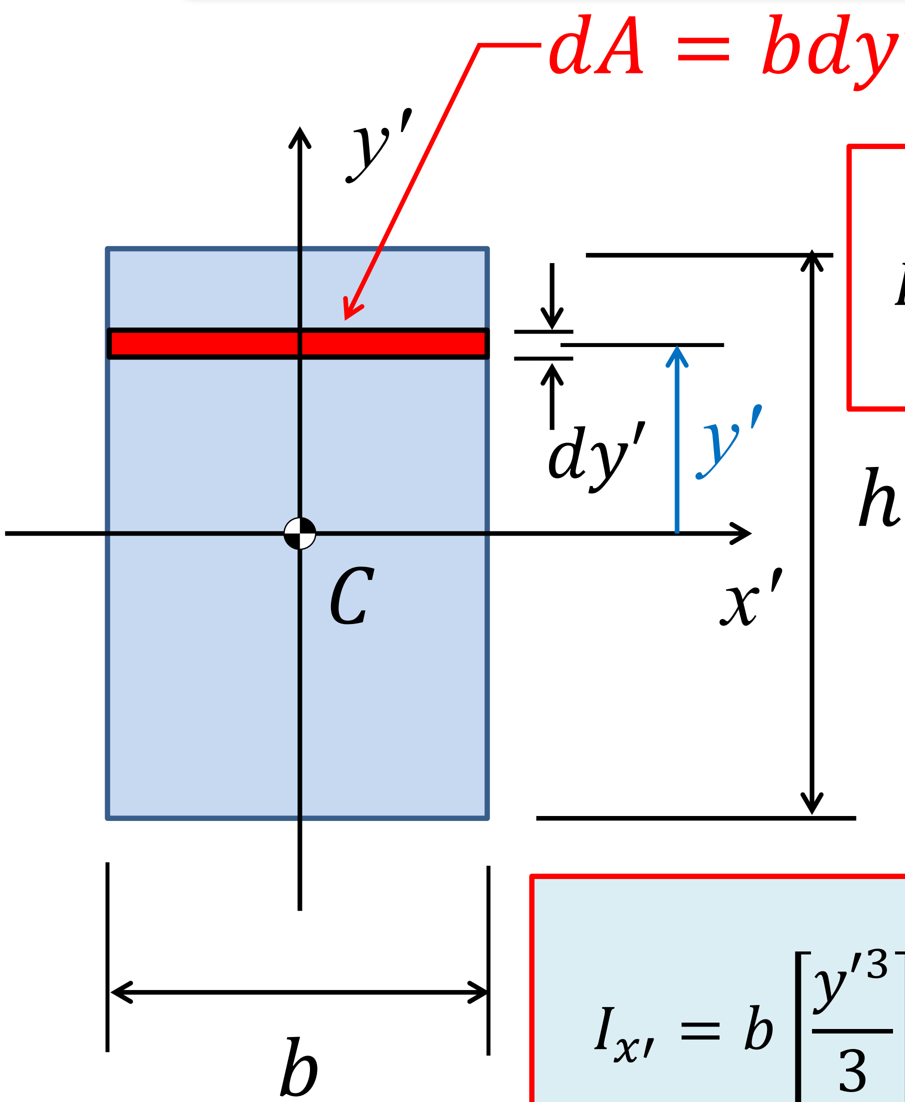
Moment of Inertia About a Centroidal Axis



Find the Moment of Inertia of the of the shaded area about the centroidal x' axis

$$I_{x'} = \iint y'^2 dA$$

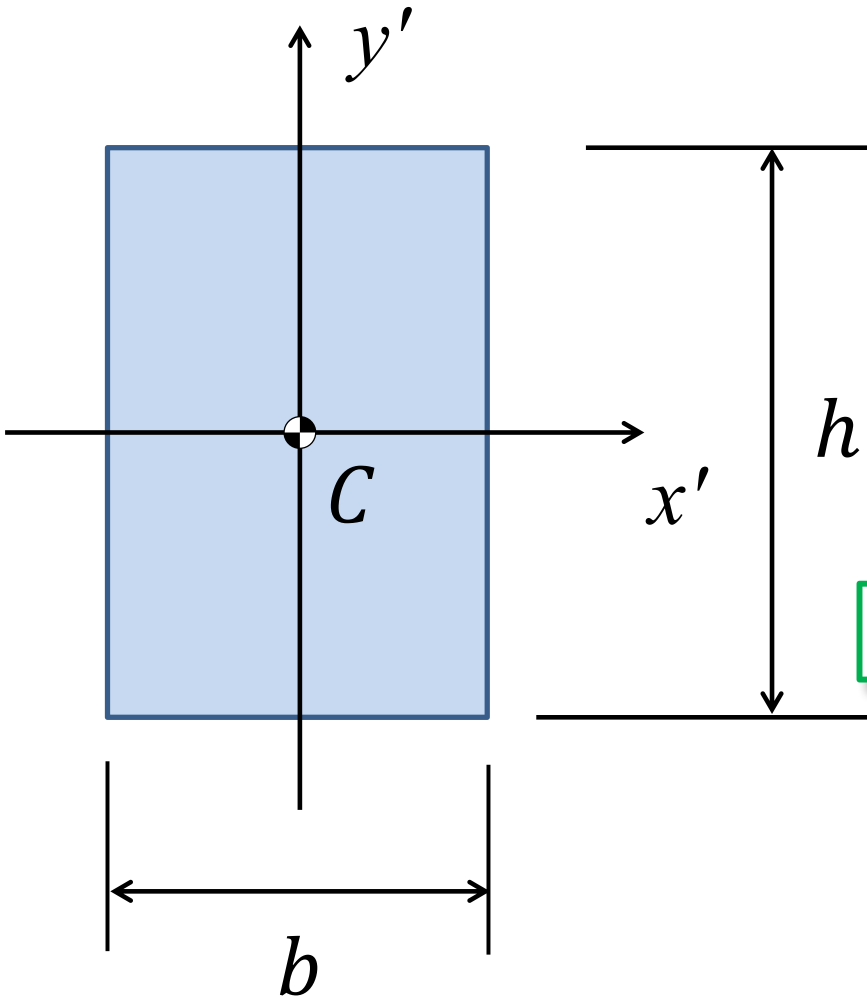
Cut Rectangle into Horizontal Strips



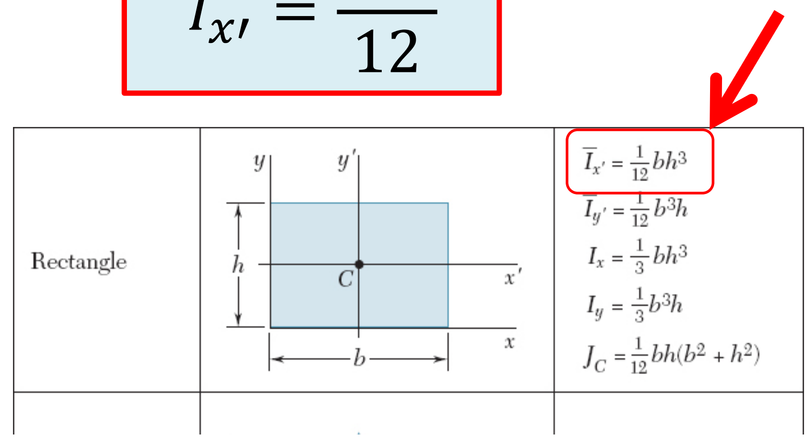
$$I_{x'} = \int_{-\frac{h}{2}}^{\frac{h}{2}} y'^2 b\,dy' = b\int_{-\frac{h}{2}}^{\frac{h}{2}} y'^2\,dy'$$

$$I_{x'} = b\left[\frac{y'^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{bh^3}{24} - \left(-\frac{bh^3}{24}\right) = \frac{bh^3}{12}$$

Moment of Inertia of a Rectangular Area About its Centriodal Axes



$$I_{x'} = \frac{bh^3}{12}$$



Agrees with the tabulated solution

Note also:

$$I_{y'} = \frac{b^3h}{12}$$